La Tour d'Hanoï: from 2 to 3 . . . [1]

Andreas M. Hinz

Ludwig-Maximilians-Universität München & Universa v Mariboru

CombinatoireS

École d'Été
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0. Paris

1. Pairs

ordered pairs $(x, y) \in V \times V =: V^2$; $|V^2| = |V|^2$

unordered pairs $\{x, y\} \in \binom{V}{2}$

(more general: $\binom{V}{k} := \{U \subset V \mid |U| = k\} \subset 2^V$; $|\binom{V}{k}| = \binom{|V|}{k}$, $|2^V| = 2^{|V|}$)

[1]

$E \subset \binom{V}{2}$ is called an *association* (on V), $G = (V, E)$ is a (*simple*) *graph*

An association E is *transitive*, iff $\forall \{x, y, z\} \in \binom{V}{3} : \ \{x, z\}, \{y, z\} \in E \Rightarrow \{x, y\} \in E$.

A transitive association is called an *equivalence* (on V).

x and y are *equivalent* ($x \approx y$) if either $x = y$ or $\{x, y\} \in E$.

$[x] := \{y \in V \mid x \approx y\}$ is the *equiset* (*equivalence class*) of x in V w.r.t. E.

Example 1. $V = \mathbb{Z}$, $E = \left\{\{m, n\} \in \binom{\mathbb{Z}}{2} \mid m \cdot n > 0\right\}$; equisets $-\mathbb{N}$, $\{0\}$, $\mathbb{N}$.

Example 2. Let $\widetilde{E} := \left\{\{x, y\} \in \binom{V}{2} \mid \exists\, x, y - \text{walk in } G\right\} \supset E$; $\widetilde{E}$ is transitive.

$G = (V, E)$ is called *complete*, iff $E = \binom{V}{2}$; *connected*, iff $\widetilde{E} = \binom{V}{2}$.

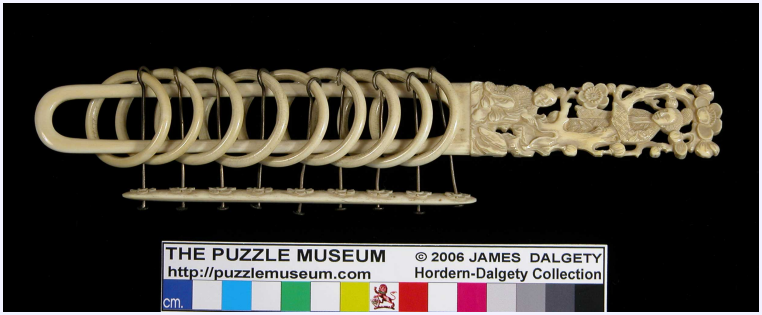
$$G \text{ complete } \Leftrightarrow G \text{ connected and } E = \widetilde{E}\,.$$

$([x]_{\widetilde{E}}, E_x)$ with $E_x = \left\{\{y, z\} \in E \mid y, z \in [x]_{\widetilde{E}}\right\}$, is called a *component* of G.

Every component of a graph G is connected. If $V \neq \emptyset$, then G is connected, iff $|V/\widetilde{E}| = 1$.

E transitive $\Leftrightarrow$ every component of G is complete.

2. The Chinese Rings (Le Baguenaudier, Меледа)



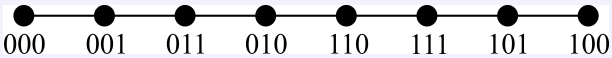
Chinese rings—jiu lian huan (trad., before 1500)

Modelling by *state graphs*

Chinese rings graph R^n ($n \in \mathbb{N}_0$ rings) $(\omega^{(k)} := 10\ldots0 \in [2]_0^k;\ k \in \mathbb{N}_0)$

$$V(R^n) = [2]_0^n,\ E(R^n) = \left\{ \left\{ \underline{s}0\omega^{(r-1)}, \underline{s}1\omega^{(r-1)} \right\} \mid r \in [n] \right\}$$

$R^n \cong P_{2^n}$



$n = 3$

$\mathrm{d}(0^n, \omega^{(n)}) = \varepsilon(0^n) = \mathrm{diam}(R^n) = 2^n - 1 =: M_n$ (*Mersenne sequence*)

For $\ell_n := \mathrm{d}(0^n, 1^n)$ we have $\ell_n + \ell_{n-1} = M_n$ (*Lichtenberg sequence*, 1769)

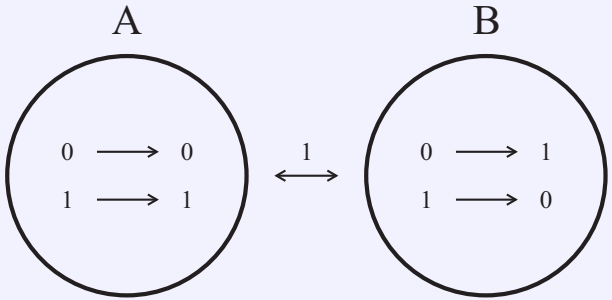
$\ell = 0, 1, 2, 5, 10, 21, 42, 85, \ldots = 0_2, 1_2, 10_2, 101_2, 1010_2, 10101_2, 101010_2, 1010101_2, \ldots$

Algorithm. To get from 1^n to 0^n, make alternating moves of ring 1 and another ring, starting with ring 1, iff n is odd; in particular, $\ell_9 = 341$.

What is $\mathrm{d}(s) := \mathrm{d}(0^n, s)$ (the *Gros weight*) for general $s \in [2]_0^n$?

Gros code automaton

$d_{r-1} = d_r \dot{\vee} s_r$



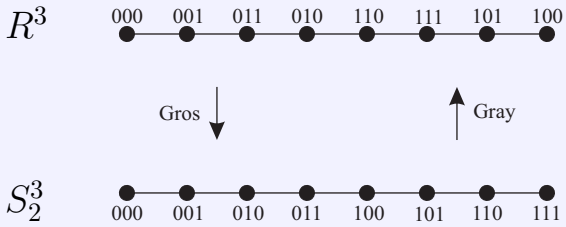
$d = \int \sigma$, $\sigma_k = s_{k+1}$

Sierpiński graphs with *base* $p \geq 2$ and *exponent* $n \in \mathbb{N}_0$: $V\left(S_p^n\right) = [p]_0^n$,

$$E\left(S_p^n\right) = \left\{ \left\{\underline{s}ij^{d-1}, \underline{s}ji^{d-1}\right\} \mid \{i,j\} \in \binom{[p]_0}{2},\ d \in [n] \right\}$$

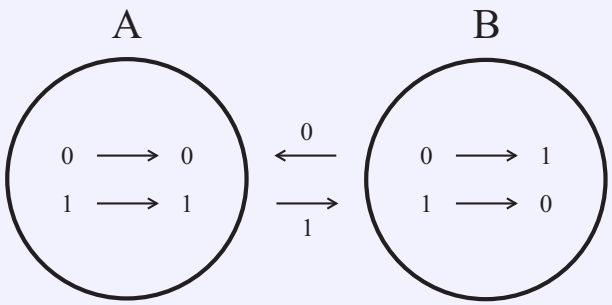
So $R^n \cong S_2^n$. $\gamma(R^n) = \left\lceil \frac{1}{3} 2^n \right\rceil$, based on the fundamental relation $M_n \bmod 3 = n \bmod 2$.

The inverse is given by the Gray code.



Gray code automaton

$$s_r = d_r \dot{\vee} d_{r-1}$$



$$\sigma = d'$$

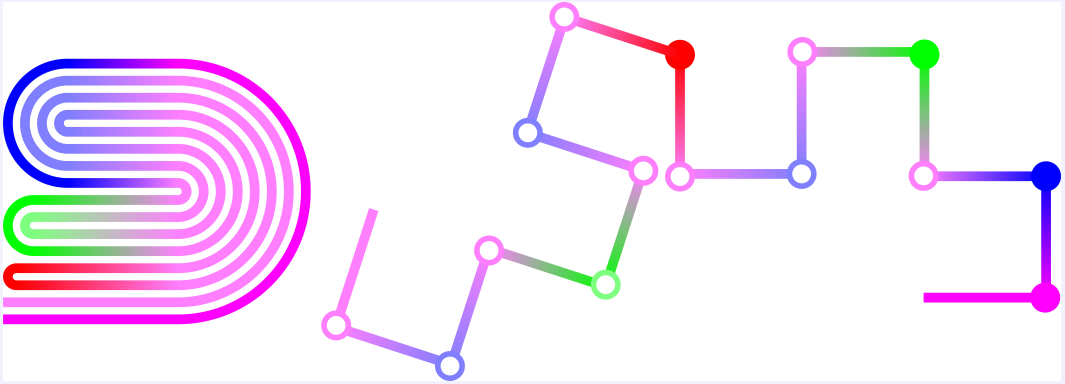
Starting in state 0^∞, the ring moved in step $k = 2^{r-1}(2\kappa - 1)$ is $g_k = r$; $\kappa \in \mathbb{N}$.

This is the *Gros sequence*

$$g = 1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5, 1, 2, 1, 3, 1, \ldots$$

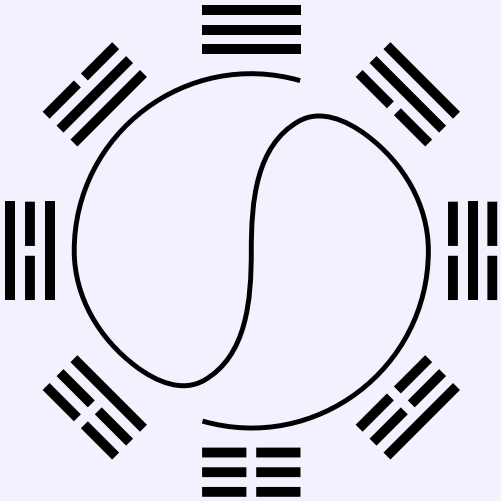
It is the *greedy (strongly) square-free sequence* over $\mathbb{N}$.

The sequence of ups (1) and downs (0) of the rings is the *paper-folding sequence* $\varphi = 1, 1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 1, 0, 0, \ldots$.

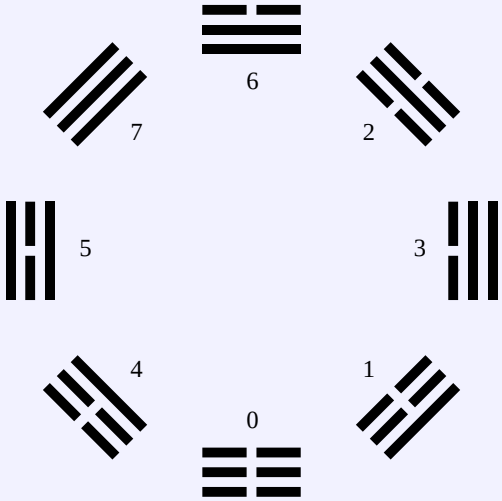


Let $\overline{g} := g \bmod 2$; then $1 - \overline{g}$ *period doubling sequence*, $\sum \overline{g}$ *Thue-Morse sequence*.

The 8 trigrams *ba gua*



Fu Xi



Gray

3. The Tower of Hanoi

La Tour d'Hanoï (Édouard Lucas, 1883)

3.0. The P0-Task (perfect to perfect)

Legal distributions of discs from $[n]$ on pegs from $[3]_0$ are coded by $s \in [3]_0^n$.

The *P0-task* is to get from 0^n to 2^n (on a shortest possible path).

Proposition 0 *The P0-task can be achieved in M_n moves.*

Proof by induction. The statement is clearly true for $n = 0$. To transfer a tower of $n+1$ discs from peg 0 to peg 2, displace the n-tower of smaller discs to peg 1 in M_n moves (by induction assumption), then disc $n+1$ to peg 2 in 1 move and finally all other discs on top of it in another M_n moves. Altogether $2M_n + 1 = M_{n+1}$ moves have been performed. $\square$

Is this *recursive algorithm* optimal or was there a hidden assumption in the previous proof? Yes and Yes!

Why does disc $n+1$ move *directly* to the goal peg?

Theorem *The optimal solution for the P0-task is unique and takes M_n moves.*

Proof by induction. The statement is clearly true for $n = 0$. Before the first move of disc $n+1$ in any solution, a tower of n discs has to be transferred from the source peg to another one, which takes at least M_n moves by induction assumption. After the last move of disc $n+1$ again an n-tower changes position from some peg to the goal peg, consuming another M_n individual disc moves. Since disc $n+1$ has to move at least once, the solution needs at least M_{n+1} steps. Therefore, by Proposition 0, first and last move of disc $n+1$ coincide, and uniqueness of the optimal solution follows from induction assumption as well. $\square$

Is there an *efficient algorithm* for the optimal solution?

Is there a *human algorithm* for the optimal solution?

The optimal solution

Observation (Raoul Olive): disc 1 moves cyclically

Moreover, the disc moved in move number k is g_k

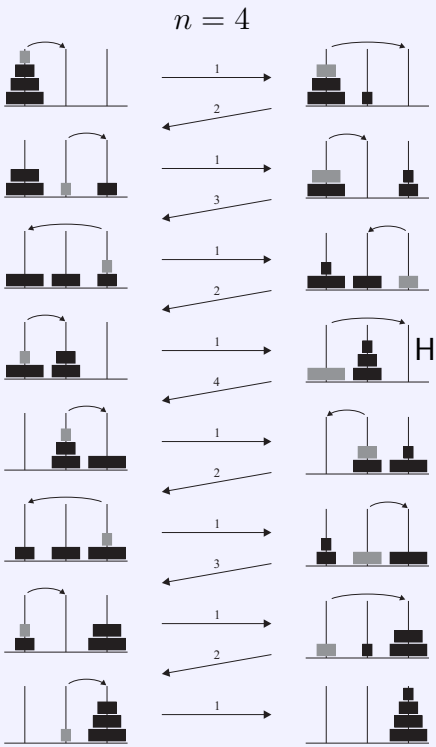
Hence, disc d moves for the first time in step 2^{d-1}

The only move of the largest disc

Call bottom of peg i "disc $n+1+i$";

then all "discs" on a peg are of alternating parity.

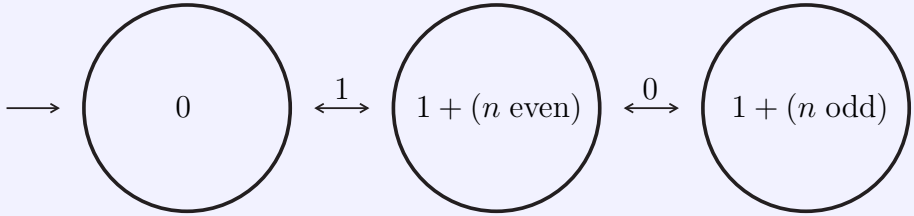
All this leads to *iterative algorithms*.



Half-way solution

What happens, if we introduce a "disc 0"?

"Disc 0" acts like a "thimble" designating the *idle peg*



idle peg automaton

yields the idle peg of move $k = \sum_{d=1}^{n} k_d \cdot 2^{d-1}$;

this, together with the *divine rule* defines the move completely.

Olive's sequence o: start in 0^∞ and let the idle peg follow the sequence $(012)^\infty$. Then

$$o = 1, \overline{2}, 0, 1, 2, \overline{0}, 1, \overline{2}, 0, \overline{1}, 2, 0, 1, \ldots = (g_\ell \bmod 2, \ell \bmod 3)\,,$$

where $(0, i) = \overline{i}$ and $(1, i) = i$.

The Olive sequence is square-free and automatic (Allouche & al., 1989ff).

parallel algorithm or *spreadsheet solution*

$d \setminus k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	**0**	**1**	**1**	**2**	**2**	**0**	**0**	**1**	**1**	**2**	**2**	**0**	**0**	**1**	**1**	**2**
2	**0**	**0**	**2**	**2**	**2**	**2**	**1**	**1**	**1**	**1**	**0**	**0**	**0**	**0**	**2**	**2**
3	**0**	**0**	**0**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**2**	**2**	**2**	**2**
4	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**2**	**2**	**2**	**2**	**2**	**2**	**2**	**2**

Another curiosity: The number of distinct distributions of discs on the intermediate peg during execution of the optimal solution for $n \in \mathbb{N}_0$ discs is F_{n+1}.

We can also solve the inverse problem. But can we trust the Brahmins?

La Tour d'Hanoï: . . . to 4 and beyond[2]

Andreas M. Hinz

Ludwig-Maximilians-Universität München & Universa v Mariboru

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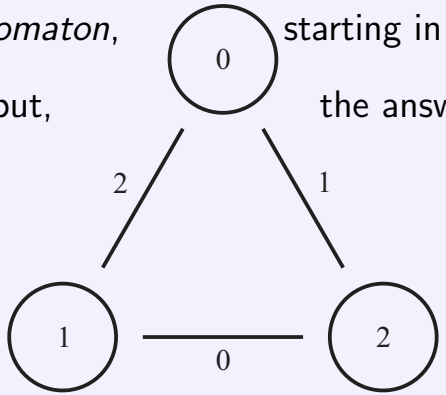
3.1. The P1-Task (regular to perfect)

Does $s \in [3]_0^n$ lie on the optimal $1^n, 2^n$-path?

Enter $s_n, \ldots, s_1$ into *P1 automaton*, starting in state 0:

iff state changes on every input, the answer is Yes!



Entering i in state j of the automaton yields $i \vartriangle j$ according to the

Cayley table (Note that $i \vartriangle j = 3 - i - j$, if $i \neq j$.)

$\vartriangle$	0	1	2
0	0	2	1
1	2	1	0
2	1	0	2

For the *P1-task* $[3]_0^n \ni s \to j^n$, the P1 automaton gives

— the disc to be moved in the optimal first move

— the idle peg $s_1 \vartriangle \cdots \vartriangle s_n \vartriangle j$ in that move and

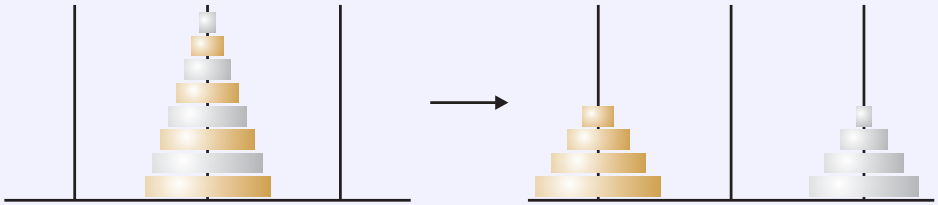
— the length $\mathrm{d}(s, j^n)$ of the (only) optimal solution.

Every $s \in [3]_0^n$ is uniquely determined by the two values $\mathrm{d}(s, i^n)$, $\mathrm{d}(s, j^n)$, and $\mathrm{d}(s, i^n) + \mathrm{d}(s, j^n) + \mathrm{d}(s, (i \vartriangle j)^n) = 2M_n$, if $i \neq j$.

More challenging tasks: sorting gold and silver!

This corresponds to a P1-task $s \to 1^n$ with $s_d = 2(d \bmod 2)$ for $d \in [n]$.

Needs $\left\lfloor \frac{5}{7} 2^n \right\rfloor = 0, 1, 2, 5, 11, 22, 45, 91, 182, \ldots$ moves.



So far, everything can be proved by induction.

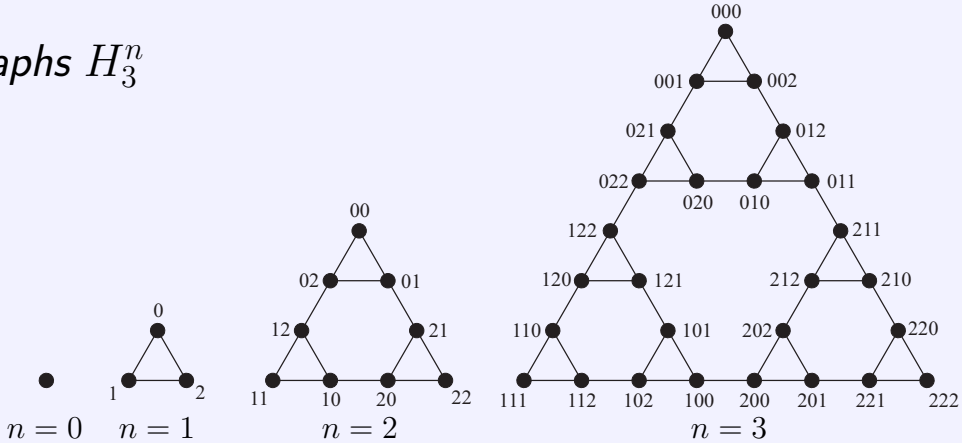
But what, for instance, if we want to switch gold and silver towers?

This is a *P2-task* $[3]_0^n \ni s \to t \in [3]_0^n$ (with $t_d = 2(1 - (d \bmod 2))$).

Surprise: there may be two optimal solutions ($01 \to 10$) and
it might even be necessary to move the largest disc twice ($011 \to 100$)!

3.2. Hanoi Graphs and the P2-Task (regular to regular)

Hanoi graphs H_3^n



Metric properties: $\mathrm{d}(0^n, 2^n) = \varepsilon(0^n) = \mathrm{diam}(H_3^n) = M_n$.

On a shortest path the largest discs moves at most twice;

there are up to two shortest paths differing by the number of LDMs.

(Hinz, 1989) $\Theta_\pm := \frac{1}{2}(5 \pm \sqrt{17})$

$$\overline{\mathrm{d}}(H_3^n) = \frac{466}{885}\, 2^n - \frac{1}{3} + \frac{6}{59}\left(2 + \frac{3}{17}\sqrt{17}\right)\left(\frac{\Theta_+}{9}\right)^n - \frac{3}{5}\left(\frac{1}{3}\right)^n + \frac{6}{59}\left(2 - \frac{3}{17}\sqrt{17}\right)\left(\frac{\Theta_-}{9}\right)^n$$

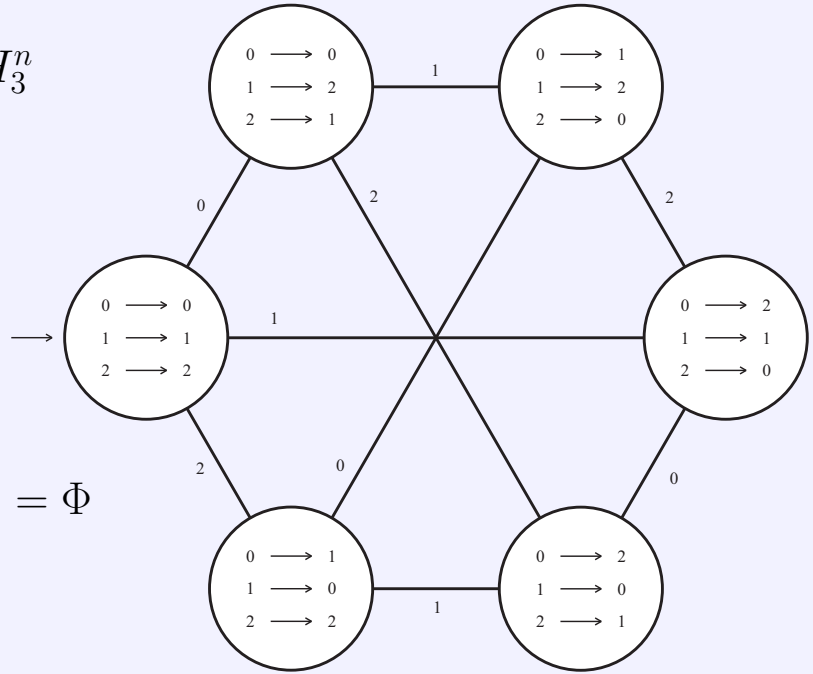
(Hinz and Schief, 1990) Average distance on the *Sierpiński triangle*

©A.M.Hinz, 2001

Floor decoration in San Clemente, Rome

W. Sierpinski, C. R. Acad. Sci. 1915–02–01 (présenté par E. Picard)

$S_3^n \cong H_3^n$



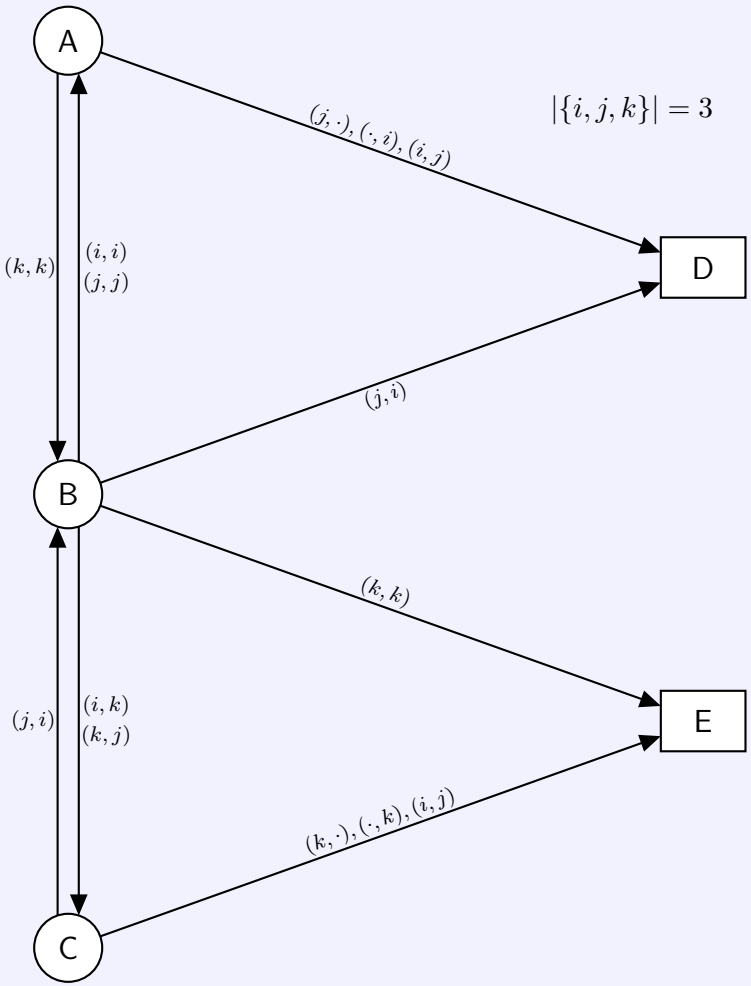
$\Phi^{-1} = \Phi$

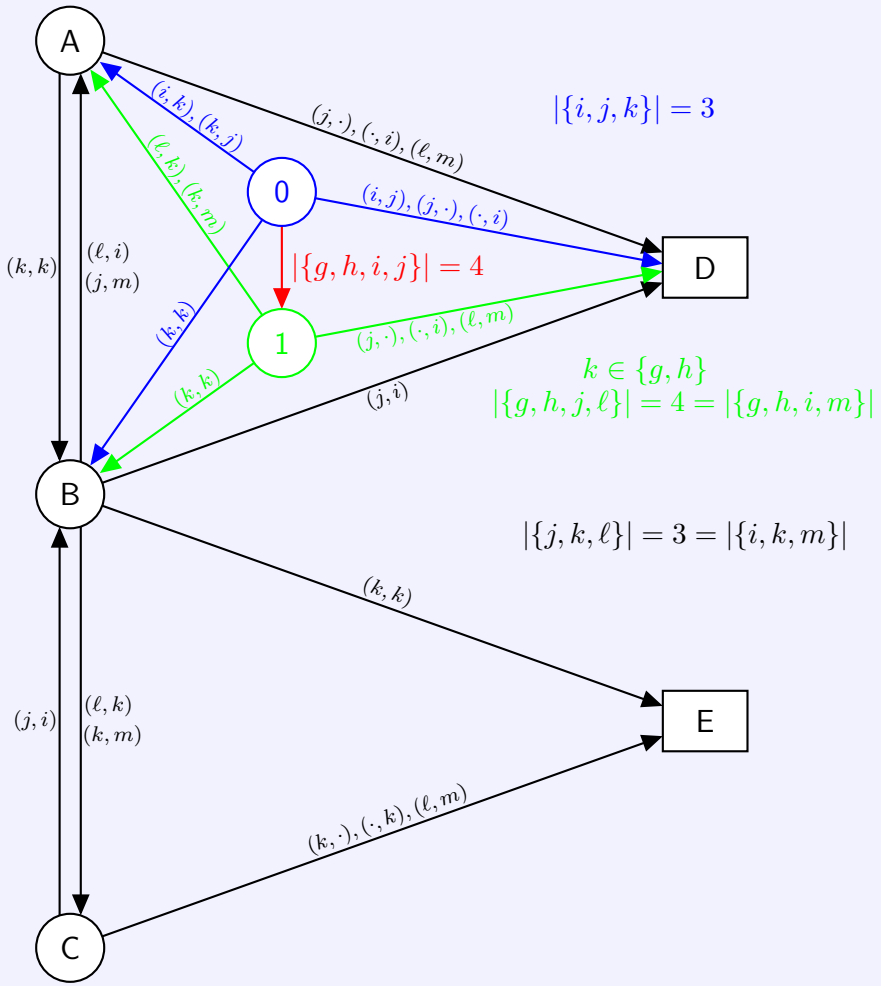
Hanoi graphs with *base* $p \geq 3$ and *exponent* $n \in \mathbb{N}_0$: $V\left(H_p^n\right) = [p]_0^n$,

$$E\left(H_p^n\right) = \left\{ \{\underline{s}i\overline{s}, \underline{s}j\overline{s}\} \mid \{i,j\} \in \binom{[p]_0}{2},\ d \in [n],\ \overline{s} \in \left([p]_0 \setminus \{i,j\}\right)^{d-1} \right\}$$

For S_p^n: P2 automata by D. Romik ($p = 3$, 2006),

A. M. Hinz and C. Holz auf der Heide (general p, 2014)


A
B
C
D
E
$|\{i,j,k\}| = 3$
$(j,\cdot),(\cdot,i),(i,j)$
(k,k)
(i,i)
(j,j)
(j,i)
(k,k)
(j,i)
(i,k)
(k,j)
$(k,\cdot),(\cdot,k),(i,j)$


A
B
C
D
E
0
1
$|\{i,j,k\}| = 3$
$|\{g,h,i,j\}| = 4$
$k \in \{g,h\}$
$|\{g,h,j,\ell\}| = 4 = |\{g,h,i,m\}|$
$|\{j,k,\ell\}| = 3 = |\{i,k,m\}|$
$(j,\cdot),(\cdot,i),(\ell,m)$
$(i,k),(k,j)$
$(\ell,k),(k,m)$
$(i,j),(j,\cdot),(\cdot,i)$
(k,k)
(ℓ,i)
(j,m)
$(j,\cdot),(\cdot,i),(\ell,m)$
(j,i)
(k,k)
(j,i)
(ℓ,k)
(k,m)
$(k,\cdot),(\cdot,k),(\ell,m)$

$$\mathrm{d}(s, j^n) = \sum_{d=1}^{n} (s_d \neq j) \cdot 2^{d-1}$$ unique solution

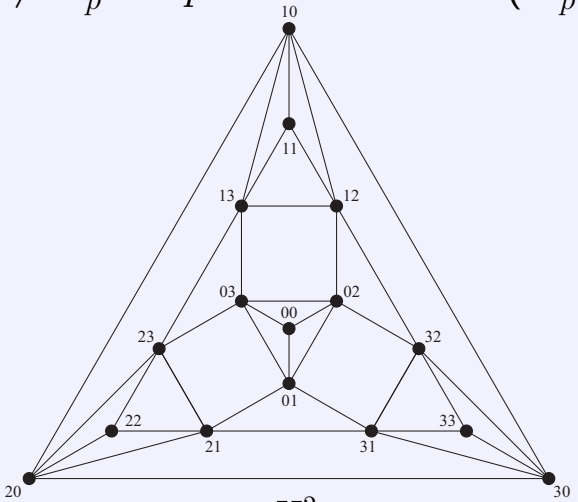
$$\mathrm{d}(is, jt) = \mathrm{d}(s, j^n) + 1 + \mathrm{d}(t, i^n)$$ A, D; unique solution, 1 LDM

or $$= \mathrm{d}(s, k^n) + 1 + 2^n + \mathrm{d}(t, k^n)$$ C, E; unique solution, 2 LDMs

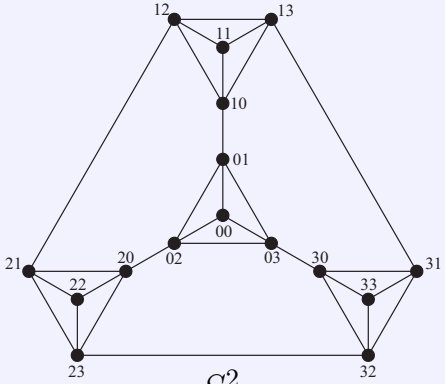
or B; 2 solutions

In S_p^n: $\mathrm{d}(0^n, (p-1)^n) = \varepsilon(0^n) = \mathrm{diam}(S_p^n) = M_n$

But: $H_p^n \ncong S_p^n$ for $p > 3$ and $n > 1$ $(H_p^1 \cong K_p \cong S_p^1)$



H_4^2



S_4^2

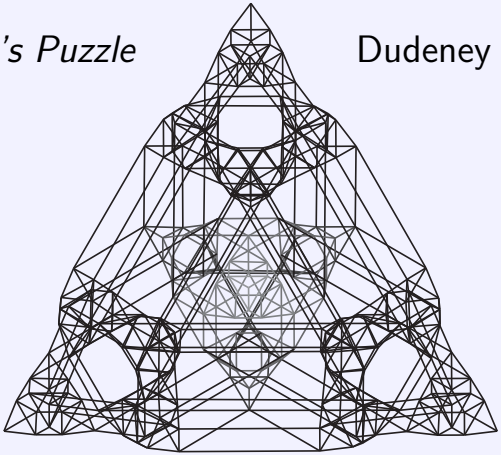
There may be up to $p-1$ LDMs (necessary) in H_p^n! ($022333 \to 300101$ in H_4^6)

4. The Reve's Puzzle

H_p^n has more complex metric properties: for $n \in \mathbb{N}$ we know

$$2n-1 \leq \mathrm{d}\left(0^n,(p-1)^n\right) \leq \varepsilon(0^n) \leq \mathrm{diam}(H_p^n) \leq 2^n-1$$

Case $p=4$: *The Reve's Puzzle* Dudeney 1902–8



Hanoi graph H_4^4

Frame-Stewart numbers are defined as

$$FS_4^0 = 0,\ \forall\, n \in \mathbb{N}:\ FS_4^n = \min\left\{2FS_4^m + M_{n-m} \mid\ m \in [n]_0\right\}$$

Theorem $\forall\, \nu \in \mathbb{N}_0,\ x \in [\nu+1]_0:\ FS_4^{\Delta_\nu + x} = (\nu - 1 + x)\, 2^\nu + 1$

Conjecture (Frame and Stewart, 1941) $\mathrm{d}(0^n, 3^n) = FS_4^n$ *in* H_4^n

Dunkel's "Lemma". After $(\nu + x)\, 2^{\nu-1}$ moves at most $\Delta_\nu + x$ discs have left peg 0.

(Korf and Felner, 2007) numerical confirmation of FSC for $n \leq 30$

Korf's Phenomenon: $ex(n) = \varepsilon(0^n) - \mathrm{d}(0^n, 3^n)$, $EX(n) = \mathrm{diam}(H_4^n) - \mathrm{d}(0^n, 3^n)$

n	12	13	14	15	16	17	18	19	20	21	22
$\mathrm{d}(0^n, 3^n)$	81	97	113	129	161	193	225	257	289	321	385
$\varepsilon(0^n)$	81	97	113	**130**	161	193	225	257	**294**	**341**	**394**
$ex(n)$	0	0	0	**1**	0	0	0	0	**5**	**20**	**9**
$EX(n)$	0	0	0	1	≥ 0	≥ 0	≥ 0	≥ 0	≥ 5	≥ 20	≥ 9

Conjecture (Korf and Felner, 2007) For any $n \geq 20$, $ex(n) > 0$.

Conjecture (Hinz et al., 2013) The function EX is (eventually strictly) monotone increasing.

$q \setminus \nu$	0	1	2	3	4	5	6	7	8	9	10
0	0	1	1	1	1	1	1	1	1	1	1
1	0	1	2	3	4	5	6	7	8	9	10
2	0	1	3	6	10	15	21	28	36	45	55
3	0	1	4	10	20	35	56	84	120	165	220
4	0	1	5	15	35	70	126	210	330	495	715
5	0	1	6	21	56	126	252	462	792	1287	2002
6	0	1	7	28	84	210	462	924	1716	3003	5005
7	0	1	8	36	120	330	792	1716	3432	6435	11440
8	0	1	9	45	165	495	1287	3003	6435	12870	24310
9	0	1	10	55	220	715	2002	5005	11440	24310	48620

The hypertetrahedral array $\Delta_{q,\nu} = \binom{q+\nu-1}{q}$

Pascal's *Arithmetical triangle*

$q \setminus \nu$	0	1	2	3	4	5	6	7	8	9	10
0	1	1	1	1	1	1	1	1	1	1	1
1	-1	0	1	2	3	4	5	6	7	8	9
2	1	1	2	4	7	11	16	22	29	37	46
3	-1	0	2	6	13	24	40	62	91	128	174
4	1	1	3	9	22	46	86	148	239	367	541
5	-1	0	3	12	34	80	166	314	553	920	1461
6	1	1	4	16	50	130	296	610	1163	2083	3544
7	-1	0	4	20	70	200	496	1106	2269	4352	7896
8	1	1	5	25	95	295	791	1897	4166	8518	16414
9	-1	0	5	30	125	420	1211	3108	7274	15792	32206

The P-array $P_{q,\nu}$

$$2P_{q,\nu+1} = P_{q,\nu} + \Delta_{q,\nu+1}$$

$q \setminus \nu$	0	1	2	3	4	5	6	7	8
0	0	1	1	1	1	1	1	1	1
1	0	1	3	7	15	31	63	127	255
2	0	1	5	17	49	129	321	769	1793
3	0	1	7	31	111	351	1023	2815	7423
4	0	1	9	49	209	769	2561	7937	23297
5	0	1	11	71	351	1471	5503	18943	61183
6	0	1	13	97	545	2561	10625	40193	141569
7	0	1	15	127	799	4159	18943	78079	297727

Dudeney's array $a_{q,\nu}$

$$a_{q,0} = 0,\ a_{0,\nu} = \Delta_{0,\nu},\ a_{q+1,\nu+1} = 2a_{q+1,\nu} + a_{q,\nu+1}\,.$$

Dudeney's algorithm uses $\quad \Delta_{h,\nu+1} = 1 + \sum_{q=1}^{h} \Delta_{q,\nu}\,.$

$$\Longrightarrow \quad \mathrm{d}_{2+h}(0^{\Delta_{h,\nu}}, 1^{\Delta_{h,\nu}}) \leq a_{h,\nu} = P_{h-1,\nu} \cdot 2^{\nu} + (-1)^{h}$$

Fundamental relations

$$a_{h,\nu} = \sum_{k=0}^{\Delta_{h,\nu}-1} 2^{\nabla_{h,k}} \,,$$

where $\nabla_{h,k} = \max\{\mu \in \mathbb{N}_0 \mid \Delta_{h,\mu} \leq k\}$ is the *hypertetrahedral root* of k, for which

$$\forall\, x \in [\Delta_{h-1,\nu+1}]_0 : \;\; \nabla_{h,\Delta_{h,\nu}+x} = \nu \,.$$

$h \setminus n$	0	1	2	3	4	5	6	7	8	9
1	0	1	3	7	15	31	63	127	255	511
2	0	1	3	5	9	13	17	25	33	41
3	0	1	3	5	7	11	15	19	23	27
4	0	1	3	5	7	9	13	17	21	25

$$\forall\, x \leq \Delta_{h-1,\nu+1} : \;\; FS_{h+2}^{\Delta_{h,\nu}+x} = a_{h,\nu} + x \cdot 2^{\nu}$$

Altogether we know

$$FS^n_{2+h} = \sum_{k=0}^{n-1} 2^{\nabla_{h,k}} =: \Phi_{h-1}(n),\ \overline{FS}^n_{2+h} = \frac{1}{2}\sum_{k=1}^{n} 2^{\nabla_{h,k}} =: \overline{\Phi}_{h-1}(n)\,.$$

To prove the

Frame-Stewart Conjecture $\qquad \mathrm{d}_p(0^n, 1^n) = FS^n_p\,,$

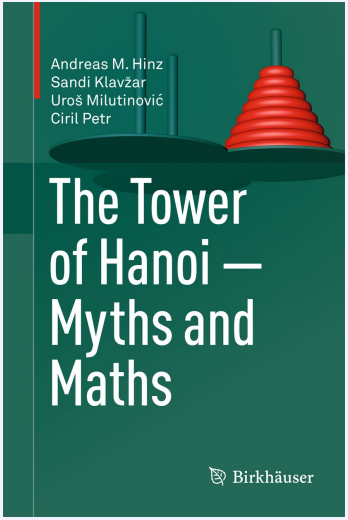
it suffices to show that

$$\forall\, t \in [h]^n :\ \mathrm{d}_{2+h}(0^n, t) \geq \overline{\Phi}_{h-1}(n)\,.$$

Note that for $p = 3$, i.e. $h = 1$, we have $t = 1^n$ and $\overline{\Phi}_0(n) = 2^n - 1$.

Thierry Bousch has published an attempt at $p = 4$, i.e. $h = 2$, in 2014.

Further reading:



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